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Abstract: We consider traffic grooming in WDM unidirectional rings with all-to-all uniform unitary traffic. We determine the minimum number of SONET/SDH add-drop multiplexers (ADMs) required when the grooming ratio is 3. In fact, using tools of design theory, we solve the equivalent edge partitioning problem: find a partition of the edges of the complete graph on n vertices (K_n) into subgraphs having at most 3 edges and in which the total number of vertices has to be minimized.

Key-words: Traffic grooming, graph, edge-partition, design theory, WDM rings.

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Minimisation du nombre d'ADM dans les anneaux WDM unidirectionnel avec un facteur de groupage 3

Résumé : Nous considérons le problème du groupage de trafic dans les anneaux WDM unidirectionnels dans le cas d'un échange total avec un trafic unitaire uniforme. Nous déterminons le nombre minimum de multiplexeurs à insertion/extractions (ADM) lorsque le facteur de groupage est 3. En utilisant des outils de la théorie des designs, nous résolvons le problème équivalent de partition d'arêtes : trouver une partition des arêtes du graphe complet à n sommets (K_n) en sous-graphes ayant au plus 3 arêtes telle que le nombre total de sommets soit minimum.

Mots-clés : Groupage, graphe, partition des arêtes, théorie des designs, anneaux WDM.

1 Introduction

Traffic grooming is the generic term for packing low rate signals into higher speed streams (see the surveys [9, 13, 15]). By using traffic grooming, one can bypass the electronics in the intermediate nodes. Typically, in a WDM network, instead of having one SONET/SDH Add Drop Multiplexer (or ADM) on every wavelength at every node, it may be possible to have ADMs only for the wavelengths used at that node (the other wavelengths being optically routed without electronic switching). The objective is either to minimize the transmission cost, in particular the number of wavelengths, or to minimize the equipment cost, in particular the total number of ADMs used in the network.

Here, we consider the particular case of unidirectional rings with static uniform symmetric all-to-all traffic. In this case, for each pair $\{i, j\}$, we associate a circle (or circuit) which contains both the request from i to j and from j to i. If each circle requires only $\frac{1}{C}$ of the bandwidth of a wavelength, we can "groom" C circles on the same wavelength. C is called the grooming ratio (or grooming factor). For example, if the request from i to j (and from j to i) is one OC-12 and a wavelength can carry an OC-48, the grooming factor is 4. Given the grooming ratio C and the size n of the ring, the objective is to minimize the total number of ADMs used, denoted $A(C, n)$, and therefore to reduce the network cost by eliminating as many ADMs as possible from the "no grooming case". For example, let $n = 4$; we have 6 circles corresponding to the 6 pairs $\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$. Without grooming, that is if we assign one wavelength per circle, we need 2 ADMs per circle; then a total of 12 ADMs are required. Suppose now that $C = 3$, that is we can groom 3 circles on one wavelength. One can groom on wavelength 1 the circles associated with $\{1,2\}, \{1,3\}, \{1,4\}$ using 4 ADMs and on wavelength 2 those associated with $\{2,3\}, \{2,4\}, \{3,4\}$ using 3 ADMs for a total of 7 ADMs.

This case of unidirectional rings with static uniform symmetric all-to-all traffic has been considered by many authors [1, 3, 8, 10, 11, 12, 16, 17, 18, 19, 20] and numerical results, heuristics and tables have been given (see for example those in [17]). This case presents the advantage of concentrating on the grooming phase (excluding the routing). It can also be applied to groom components of more general connections than two opposite pairs into wavelengths or more general classes. These components are called circles [3, 20] or circuits [17] or primitive rings [6, 7].

In [1] it is noted that the problem of minimizing the number of ADMs for the unidirectional ring C_n, with a grooming factor C, can be expressed as follows: partition the edges of the complete graph on n vertices (K_n) into W subgraphs B_λ, $\lambda = 1, 2, \ldots, W$, having $|E(B_\lambda)|$ edges and $|V(B_\lambda)|$ vertices, with $|E(B_\lambda)| \leq C$ and where $\sum_{\lambda=1}^{W} |V(B_\lambda)|$ has to be minimized (the edges of K_n correspond to the circles, the subgraphs B_λ correspond to the wavelengths and a vertex of B_λ corresponds to an ADM). In [1] various results are given using tools of design theory [5] and they improve and unify all the preceding results in the literature. Note that design theory was also used in [6, 7] for a slightly different problem with $C = 8$, as they consider bidirectional rings and 4 requests are grouped in a circle.

Here we completely solve the case $C = 3$. This case is easy to solve when there exists a partition of K_n into K_3's (triangles), as K_3 is the graph with 3 edges having the minimum number of vertices. For that we can use the existence of Steiner Triple Systems or $(n, 3, 1)$-designs which can be stated as follows (for a proof see [2]).

Theorem 1 (Steiner's theorem) *For any $n \equiv 1, 3 \pmod 6$, the edges of K_n can be partitioned into K_3's.*

Note that the problem we consider here is different from the problems of design theory in which one looks for a partition of the edges into isomorphic subgraphs and such a partition exists only for some values of n. For other values one can think to use results on packings or coverings to solve our problem. For example, for $C = 3$ and $n \not\equiv 1$ or $3 \pmod 6$, one can think that the best solution is obtained by taking as many K_3's as possible, but it does not necessarily lead to an optimal solution. Consider for example K_6. It can be partitioned into the 4 triangles $(1, 2, 3), (1, 4, 5), (2, 4, 6), (3, 5, 6)$ plus the 3 edges 1-6, 2-5, 3-4. So, altogether we have 5 subgraphs and 18 ADMs. However, we can also partition K_6 into the 3 K_3's $(1, 2, 3), (2, 4, 5), (3, 5, 6)$, the star $K_{1,3}$ with edges 1-4,1-5,1-6, and the path P_4 with edges 2-6,6-4,4-3. This solution uses 5 subgraphs and 17 ADMs. Similarly, if we use a covering of the edges of K_6 by 6 K_3's and delete the edges covered twice, we will have to use 6 subgraphs with 3 vertices and altogether 18 vertices. More generally, when n is even, a covering of K_n by K_3's needs $\left\lceil \frac{n^2}{6} \right\rceil$ triangles and 3 times more ADMs; but we will show that the optimal solution uses about $\left\lceil \frac{n}{4} \right\rceil$ fewer ADMs.

Here, we determine the exact value of $A(3, n)$, denoted by $A(n)$. The main theorem can be stated as follows:

Theorem 2

(*i*) *When n is odd, $A(n) = n(n-1)/2 + \epsilon$, where $\epsilon = 0$ if $n \equiv 1$ or $3 \pmod 6$, and $\epsilon = 2$ if $n \equiv 5 \pmod 6$;*

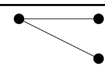
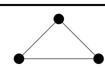
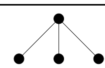
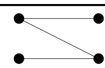
(*ii*) *When n is even, $A(n) = n(n-1)/2 + \left\lceil \frac{n}{4} \right\rceil + \epsilon$, where $\epsilon = 1$ if $n \equiv 8 \pmod{12}$, and $\epsilon = 0$ otherwise.*

Furthermore our solution uses the minimum number of subgraphs (wavelengths) possible, that is $\left\lceil \frac{n(n-1)}{6} \right\rceil$; therefore for $C = 3$, the conjecture of [3] that the minimum number of ADMs can be achieved with the minimum number of wavelengths is true (in [1] it is shown that the conjecture is false for many values of C, the first one being $C = 7$).

2 Notation

As we mentioned in the introduction, we want to partition the edges of K_n into subgraphs with at most 3 edges and to minimize the total number $A(n)$ of vertices in such a partition.

Here are the possible connected subgraphs with at most 3 edges:

Name	Class	# vertices	# odd degree vertices
E		2	2
P_3		3	2
K_3		3	0
$K_{1,3}$		4	4
P_4		4	2

For a given partition P of the edges of K_n, we denote by a_1, a_2, a_3, b_3, c_3 the number of subgraphs of type respectively E, P_3, K_3, $K_{1,3}$, P_4. By counting the number of edges of K_n we have:

$$a_1 + 2a_2 + 3a_3 + 3b_3 + 3c_3 = n(n-1)/2 \tag{1}$$

The sum of the number of vertices of the subgraphs in the partition P is denoted $A(P)$. Thus

$$A(P) = 2a_1 + 3a_2 + 3a_3 + 4b_3 + 4c_3 = n(n-1)/2 + a_1 + a_2 + b_3 + c_3 \tag{2}$$

Finally, following the definition of $A(n)$ given in the introduction, we have $A(n) = \min\{A(P) \ : \ P \text{ is a partition of } K_n\}$.

3 Lower bounds

In this section we prove that $A(n)$ has at least the value given in Theorem 2. Let P be any partition of the edges of K_n.

Case (i): if $n \equiv 1,3 \pmod 6$, equation (2) gives immediately $A(P) \geq n(n-1)/2$. Suppose now that $n \equiv 5 \pmod 6$. Thus $n(n-1)/2 \equiv 1 \pmod 3$. Then equation (1) modulo 3 gives $a_1 + 2a_2 \neq 0$.

Suppose that $a_1 + a_2 + b_3 + c_3 = 1$. Note that the subgraphs E, P_3, $K_{1,3}$ and P_4 have vertices with odd degree. As every vertex in K_n has even degree, we have a contradiction, thus $a_1 + a_2 + b_3 + c_3 \geq 2$.

Then by equation (2), $A(P) \geq n(n-1)/2 + 2$.

Case (ii) : As n is even, the degree of each vertex of K_n is odd, so every vertex must be an odd degree vertex of at least one subgraph; but the number of odd degree vertices of E, P_3, K_3, $K_{1,3}$, and P_4 are 2,2,0,4,2 respectively; thus we have the following additional inequality:

$$2a_1 + 2a_2 + 4b_3 + 2c_3 \geq n \tag{3}$$

From (3) we deduce that

$$4(a_1 + a_2 + b_3 + c_3) \geq n + 2(a_1 + a_2 + c_3)$$

So

$$a_1 + a_2 + b_3 + c_3 \geq \left\lceil \frac{n}{4} \right\rceil$$

which gives the result for the cases $n \not\equiv 8 \pmod{12}$.

Now if $n \equiv 8 \pmod{12}$, then $n(n-1)/2 \equiv 1 \pmod 3$ and thus equation (1) modulo 3 gives the additional constraint $a_1 + 2a_2 \geq 1$. Thus we have $4(a_1 + a_2 + b_3 + c_3) \geq n + 1$ and $A(P) \geq n(n-1)/2 + n/4 + 1$ as required.

4 Upper bounds

Let $p_1, p_2, \ldots, p_l$ be some nonnegative integers; the *complete multipartite graph with class sizes* $p_1, p_2, \ldots, p_l$, denoted $K_{p_1,p_2,\ldots,p_l}$ is defined to be the graph with vertex set $P_1 \cup P_2 \cup \ldots \cup P_l$ where $|P_i| = p_i$, and two vertices $x \in P_i$ and $y \in P_j$ are adjacent if and only if $i \neq j$. For $t > 0$, we denote $K_{g\times t}$ (resp. $K_{g\times t,u}$) by $K_{g,g,\ldots,g}$ (resp. $K_{g,g,\ldots,g,u}$) where g occurs t times.

Using terminology of design theory, the existence of a partition of the edges of $K_{p_1,p_2,\ldots,p_l}$ into K_k is equivalent to the existence of a *k-GDD (group divisible design) with group sizes* $p_1, p_2, \ldots, p_l$, also known as *k-GDD of type* $g_1^{a_1} g_2^{a_2} \ldots g_s^{a_s}$, where there are a_i values of the p_j's equal to g_i.

The following theorem of Colbourn [4] (see also [14, Theorem 1.24]) will be used repeatedly:

Theorem 3 *Let g, t and u some nonnegative integers. $K_{g\times t,u}$ can be decomposed into K_3's if and only if the following conditions are all satisfied:*

(i) *if $g > 0$ then $t \geq 3$, or $t = 2$ and $u = g$, or $t = 1$ and $u = 0$, or $t = 0$;*

(ii) *$u \leq g(t-1)$ or $gt = 0$;*

(iii) *$g(t-1) + u \equiv 0 \pmod 2$ or $gt = 0$;*

(iv) *$gt \equiv 0 \pmod 2$ or $u = 0$;*

(v) *$g^2 t(t-1)/2 + gtu \equiv 0 \pmod 3$.*

We now prove the upper bounds in Theorem 2. Actually we give a stronger result, by exhibiting the classes of the decomposition.

Theorem 4 *Let $n \geq 2$. There exists a partition of K_n using*

1. *if $n \equiv 1, 3 \pmod 6$, $\frac{n(n-1)}{6}$ K_3;*
2. *if $n \equiv 5 \pmod 6$, $\frac{n(n-1)-8}{6}$ K_3 and 2 P_3;*
3. *if $n \equiv 0, 4 \pmod{12}$, $\frac{n(n-1)}{6} - \frac{n}{4}$ K_3 and $\frac{n}{4}$ $K_{1,3}$;*
4. *if $n \equiv 2, 8 \pmod{12}$, $\frac{n(n-1)-2}{6} - \lceil \frac{n-2}{4} \rceil$ K_3, $\lceil \frac{n-2}{4} \rceil$ $K_{1,3}$ and 1 E;*
5. *if $n \equiv 6, 10 \pmod{12}$, $\frac{n(n-1)}{6} - \frac{n+2}{4}$ K_3, $\frac{n-2}{4}$ $K_{1,3}$ and 1 P_4.*

Proof.

First case : n odd.

If $n \equiv 1, 3 \pmod 6$, the result is exactly Theorem 1. Suppose now that $n \equiv 5 \pmod 6$.

First we deal with the cases $n = 5$ and $n = 11$:

- K_5 can be decomposed into 2 K_3's $(1, 2, 3)$, $(1, 4, 5)$, and 2 P_3's with edges 4-2,2-5 and 4-3,3-5;
- K_{11} is the union of a K_5 and a $K_{1\times 6,5}$. This K_5 can be decomposed as seen above into 2 K_3's and 2 P_3's. By Theorem 3 (with $g = 1$, $t = 6$ and $u = 5$), $K_{1\times 6,5}$ can be decomposed into 15 K_3's. So K_{11} can be decomposed into 17 K_3's and 2 P_3's.

Now for $n \geq 17$ and $n = 6p + 5$ $(p \geq 2)$, K_{6p+5} is the union of $2p$ K_3's, 1 K_5 and a $K_{3\times 2p,5}$. But K_5 can be decomposed into 2 K_3's and 2 P_3's, and $K_{3\times 2p,5}$ can be decomposed into K_3's by Theorem 3 with $g = 3$, $t = 2p$ and $u = 5$.

Second case : n even.

First, we deal with the following small cases:

- case $n = 2$: trivial;
- case $n = 4$: K_4 can be decomposed into the K_3 $(1, 2, 3)$ and the $K_{1,3}$ with edges 4-1,4-2,4-3;
- case $n = 6$: as stated in the introduction, K_6 can be decomposed into the P_4 with edges 2-6,6-4,4-3, the $K_{1,3}$ with edges 1-4,1-5,1-6, and the K_3's $(1, 2, 3)$, $(2, 4, 5)$, $(3, 5, 6)$;
- case $n = 8$: take a decomposition of K_7 in K_3's, and connect an additional vertex ∞ to all vertices of the K_7. The 7 edges incident to ∞ can be decomposed in 2 $K_{1,3}$'s and one E;

- case $n = 10$: K_{10} can be decomposed into the $K_{1,3}$ with edges 9-2,9-4,9-6, the $K_{1,3}$ with edges 10-1,10-3,10-5, the P_4 with edges 8-9,9-10,10-7, and the following K_3's:

$$\begin{array}{ccc} (1,5,6) & (1,2,8) & (1,3,9) \\ (2,6,7) & (2,3,5) & (2,4,10) \\ (3,7,8) & (3,4,6) & (5,7,9) \\ (4,8,5) & (4,1,7) & (6,8,10) \end{array}$$

Now let $n = 4t + u$ with $t \equiv 0 \pmod 3$, $t \geq 3$ and $u = 0, 2, 4, 6, 8$ or 10. K_n can be decomposed into t K_4's, a K_u and a $K_{4\times t,u}$. By Theorem 3 with $g = 4$, $K_{4\times t,u}$ can be decomposed into K_3's, except when $t = 3$ and $u = 10$, i.e. $n = 22$, for which the condition (ii) is not satisfied. Now for $n \neq 22$, each K_4 can be decomposed into a K_3 and a $K_{1,3}$, and K_u can be decomposed as shown above. Hence we get a partition of K_n into the required number of K_3's, $K_{1,3}$'s, E and P_4.

Finally for $n = 22$, K_{22} can be decomposed into 4 K_4's, a K_6 and a $K_{4\times 4,6}$. Each K_4 can be decomposed into a K_3 and a $K_{1,3}$. The K_6 can be decomposed as seen above into a P_4, a $K_{1,3}$, and 3 K_3's. By Theorem 3 with $g = t = 4$ and $u = 6$, $K_{4\times 4,6}$ can be decomposed into K_3's. Thus we get a partition of K_{22} into a P_4, 5 $K_{1,3}$'s and some K_3's, as required.

Note that, as mentioned in the introduction, our solution uses the minimum number of subgraphs possible, that is $\left\lceil \frac{n(n-1)}{6} \right\rceil$, showing that for $C = 3$, the conjecture of [3] is true. ■

5 Conclusion

In this article, we have determined the minimum number of SONET add-drop multiplexers (ADMs) required with a grooming ratio 3 in unidirectional WDM rings with all-to-all uniform unitary traffic. We have also shown that this minimum number is attained with a minimum number of wavelengths. The same ideas can be used to determine the minimum number $A(C, n)$ for larger values of C. For $C = 4$ an optimal solution can be obtained easily as we can partition the edges of K_n into C_4's and $K_3 + E$. Therefore we have $A(4, n) = n(n-1)/2$ with the minimum number of wavelengths, a result obtained also in [12]. We have also obtained partial results for the cases $C = 5$ and $C = 6$. In the latter case we can use results on 4-GDD but they are not sufficient for all the congruence classes.

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